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Trapping of passive tracers in a point vortex system

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Abstract. The advection of passive markers in the flow generated by two point vortices in a disc is studied. This Hamiltonian dynamics is investigated by means of Poincaré sections, via a set of appropriate canonical transformations. As is shown by numerical simulations, the existence of islands of regular motion around the vortices appears to be a generic feature of this system. The width of these islands is found to be proportional to the minimal distance attained by the vortices, bridging Lagrangian and Eulerian characteristics of the motion. Attention is drawn to the relevance of these results for all two-dimensional flows ruled by strong vorticity concentrations.

1. Introduction

The advection of passive tracers in non-stationary flows has been investigated intensively because of its implications on both applied and fundamental research. In general, the Lagrangian motion of the advected particles is not trivially related to the Eulerian velocity field and in recent years it has been shown that it is possible to have highly unpredictable Lagrangian motion (chaotic and mixing) even with a laminar and completely predictable Eulerian flow. On the other hand, situations have been found in which a chaotic, or even turbulent, velocity field may generate regular Lagrangian behaviour (in the sense that two close advected particles do not separate at an exponential rate). Such a situation was found to hold inside the cores of coherent vortices in two-dimensional incompressible turbulence [1], making the phenomenon extremely important for geophysical flows where, under certain conditions, big vortices dominate the dynamics. It is thus relevant to investigate in detail the Lagrangian behaviour of advected particles in simplified models, like point vortex models, which are able to reproduce some features of 2D turbulent flows.

Point vortex systems are few-degrees-of-freedom singular solutions of the two-dimensional incompressible Euler equation which share some basic features with two-dimensional turbulence [2, 3]. For two-dimensional incompressible flows, the velocity field is determined by the scalar stream function $\psi(\boldsymbol{x}, t)$, and the advection equations are a Hamiltonian system

$$\dot{x} = \frac{\partial \psi}{\mathrm{d}y} \qquad \dot{y} = -\frac{\partial \psi}{\mathrm{d}x} \tag{1}$$

with the stream function as a Hamilton function. General results on Hamiltonian systems thus guarantee that stationary Eulerian flow (in some reference frame) always generates regular Lagrangian motion, whose trajectories follow the streamlines. With a time-dependent stream function the passive particle motion can be chaotic even with regular (quasi-periodic) Eulerian flow.

In the present paper we study regular and chaotic motion of passive particles advected by the velocity field generated by two point vortices in a circular domain. Our system is conceptually equivalent to the $3 + 1$ problem in the free plane [4]. We show that vortex motion is always integrable and generates a quasi-periodic velocity field. The passive motion is governed by a time-dependent one-degree-of-freedom Hamiltonian and is investigated by means of Poincaré sections. The existence of regions of regular motion is guaranteed by general results on Hamiltonian dynamics, as has already been observed in similar problems [5–8]. We find particularly robust regular islands around vortices which exist irrespectively of the vortex motion. This fact suggests that regular islands are not related to the detailed properties of the Eulerian flow, but simply to the existence of strong concentrations of vorticity. The dimensions of regular islands are found to be proportional to the minimal vortex separation, a quantity which is analytically computable in our simple system.

The remainder of this paper is organized as follows. In section 2 we introduce the point vortex model for an inviscid flow in a circular domain and the canonical transformation for reducing the dimensionality of the system. Section 3 is devoted to the classification of two-vortex motion. In section 4 we investigate numerically the Poincaré maps of passive particle trajectories in the field generated by two point vortices and we present some numerical evidence for the main result of the present paper, i.e. regular Lagrangian motion in vortex cores. A summary and conclusions are given in section 5.

2. Point vortex model

In order to investigate the asymptotic nature of the regular islands around vortices, we restrict our study to one of the simplest systems of point vortices which can generate chaotic advection, i.e. two identical vortices on the disk.

The point vortex system is a simple model of planar flow in which the vorticity is represented by a singular distribution of line filaments perpendicular to the flow plane. The dynamical variables are the vortex coordinates and the dynamics is governed by a set of ordinary differential equations descending from the Euler equations for the continuous fluid. The point vortex model is thus a natural reduction of Euler equations to a finite number of degrees of freedom. Dynamical properties of point vortex systems have been studied by several authors interested in their chaotic motion and their connection with two-dimensional turbulence [9, 10]; see [11] for a review. The Hamiltonian theory for vortex motion inside a bounded domain was developed many years ago [12] for several boundaries.

Here we are interested in the motion inside the unitary disk D. This particular system has been chosen because it allows us to reduce the number of degrees of freedom and to investigate numerically the phase space by means of a two-dimensional Poincaré map.

The stream function generated by N point vortices of coordinates $(x_i = r_i \cos\theta_i, y_i = r_i \sin\theta_i)$ on the unitary disk D is given by

$$\psi(r,\theta) = -\frac{1}{4\pi}\sum_i^N \Gamma_i \log\left[\frac{r^2 + r_i^2 - 2rr_i\cos(\theta-\theta_i)}{1 + r^2r_i^2 - 2rr_i\cos(\theta-\theta_i)}\right] \tag{2}$$

where Γ_i is the constant circulation of the ith vortex.

The velocity field generated in a point (x, y) is then computed according to

$$u(x,y) = \frac{\partial\psi}{\partial y} \qquad v(x,y) = -\frac{\partial\psi}{\partial x} \tag{3}$$

which show that the motion of a passive particle is governed by a Hamiltonian dynamics with Hamilton function ψ.

Kirchhoff first recognized that the motion of point vortices is itself governed by Hamiltonian dynamics. The Hamilton function is derived by contracting the stream function with the singular vorticity distribution (adding a boundary condition term and removing the self-energy) to obtain

$$H = -\frac{1}{4\pi}\sum_{i>j}\Gamma_i\Gamma_j \log\left[\frac{r_i^2 + r_j^2 - 2r_i r_j \cos\theta_{ij}}{1 + r_i^2 r_j^2 - 2r_i r_j \cos\theta_{ij}}\right] + \frac{1}{4\pi}\sum_{i=1}^{N}\Gamma_i^2 \log(1 - r_i^2) \tag{4}$$

where we used the notation $\theta_{ij} = \theta_i - \theta_j$ for the polar angle between vortices i and j.

The canonical conjugated variables for (4) are the scaled coordinates $(\Gamma_i x_i, y_i)$, the phase space is thus essentially N times the configuration space D. The motion of the ith vortex is governed by the first order differential equations

$$\dot{x}_i = \frac{1}{\Gamma_i}\frac{\partial H}{\partial y_i} \qquad \dot{y}_i = -\frac{1}{\Gamma_i}\frac{\partial H}{\partial x_i}\,. \tag{5}$$

The Hamiltonian (4) is invariant under rotations in the configuration space, leading to the angular momentum as a second conserved quantity

$$L^2 = \sum_{i=1}^{N}\Gamma_i r_i^2\,. \tag{6}$$

From general results of Hamiltonian mechanics [13], a system of two point vortices in a disk is thus always integrable, while we should expect chaotic motion for $N > 2$ vortices.

In the following we will consider $N = 3$, which is the simplest case which can generate chaotic behaviour. Circulations are chosen to be $\Gamma_1 = \Gamma_2 = 1$ and $\Gamma_3 = \Gamma$. The inspection of (2)–(5) shows that a passive particle is formally equivalent to a point vortex with $\Gamma = 0$ (the passive particle does not carry circulation): this property allows us to consider the generic case $\Gamma_3 = \Gamma$ and to take the limit $\Gamma \to 0$ at the end of the computation. Thus Lagrangian motion in the point vortex system arises as a vanishing limit for the circulation of one vortex, as for the restricted problem in celestial mechanics (for a critical discussion of this limit see [14]).

The three-degree-of-freedom Hamiltonian (4) is reduced by using a set of canonical transformations described in the appendix. For $\Gamma = 0$, vortex motion is ruled by the (integrable) one-degree-of-freedom Hamiltonian

$$\begin{aligned} H_v(P_1, Q_1, P_2) = &-\frac{1}{4\pi}\log\left[\frac{2P_2 - 4\sqrt{P_1(P_2 - P_1)}\cos Q_1}{1 + 4P_1(P_2 - P_1) - 4\sqrt{P_1(P_2 - P_1)}\cos Q_1}\right] \\ &+\frac{1}{4\pi}\log[1 - 2P_1] + \frac{1}{4\pi}\log[1 - 2(P_2 - P_1)] \end{aligned} \tag{7}$$

in which $P_2 = L^2/2$ is a constant of motion. The action variable $P_1 = r_1^2/2$ is the squared radial coordinate of the first vortex, while $Q_1 = \theta_{12}$ is the relative angle between vortices. Vortex motion in the (P_1, Q_1) plane will be investigated in the following section.

The motion of the passive particle is governed by the Hamiltonian for the conjugated variables (P, Q)

$$\begin{aligned} H_p(P, Q; t) = &-\frac{1}{4\pi}\log\left[\frac{2(P_1 + P) - 4\sqrt{P_1 P}\cos(Q_1 - Q)}{1 + 4P_1 P - 4\sqrt{P_1 P}\cos(Q_1 - Q)}\right] \\ &-\frac{1}{4\pi}\log\left[\frac{2(P_2 - P_1) + 2P - 4\sqrt{(P_2 - P_1)P}\cos Q}{1 + 4(P_2 - P_1)P - 4\sqrt{(P_2 - P_1)P}\cos Q}\right] \end{aligned} \tag{8}$$

which depends on time through $P_1(t)$ and $Q_1(t)$. Here the action variable $P = r_3^2/2$ is the radial coordinate of the passive marker and $Q = \theta_{32}$ is its angle with respect to the second vortex.

3. Eulerian motion of two vortices

Since we want to test that trapping of passive particles by vortices has features which are rather insensitive to vortex motion, we explore the different behaviours of the dynamical system specified by Hamiltonian (7). The time-independent H_v is a one-degree-of-freedom Hamiltonian with the angular momentum $2P_2$ as a parameter and relative vortex motion is thus described by the isolines of H_v in the two-dimensional phase space with coordinate (P_1, Q_1). The structure in the phase space changes according to the value of angular momentum; the only constant property is that fixed points are located on $Q_1 = 0$ (corresponding to the configuration of the vortices on a line with the origin). To describe the phase space we make use of the new canonical conjugated variables

$$X(t) = \sqrt{2P_1}\cos(Q_1) = \frac{x_1x_2 + y_1y_2}{r_2}$$
$$Y(t) = \sqrt{2P_1}\sin(Q_1) = \frac{y_1x_2 - x_1y_2}{r_2} \qquad (9)$$

whose trajectories will be plotted at different values of the angular momentum.

Canonical variables (9) physically represent the Cartesian coordinates of vortex 1 in a reference frame which rotates with vortex 2 (see the appendix). For a given L^2 the motion of vortices is confined to the hypersphere defined by (6). Together with the energy conservation (4) this reduces the phase space to the two-dimensional surface (X, Y) which is, by (9), a 2D projection of the hypersphere. For $L^2 \leqslant 1$, conservation of (6) implies $X^2+Y^2 \leqslant L^2$ and for $L^2 > 1$, from the boundedness of the domain, we have $X^2+Y^2 \leqslant 1$.

A complete classification of the motion of two vortices is quite long and not essential for our purpose. In general, the number of fixed points and the topology of the separatrices change according to the value of the angular momentum. All the fixed points are located on the $Y = 0$ line (a part a singular case for $L^2 = 1$) which represents vortex positions collinear with the origin.

For any value of the angular momentum, there exists a singular point (SE) at $(X = \sqrt{L^2/2}, Y = 0)$ corresponding to vortices located in the same position. The system reduces to a single vortex of circulation $\Gamma = 2$ which rotates in the field induced by its own image. This is obviously a stable configuration, but it does not correspond to an elliptic point in the phase space of two vortices because here the Hamiltonian becomes singular.

- $L^2 < L_1^2$. There exists only one fixed point E located at $X = -\sqrt{L^2/2}$ (figure 1). It corresponds to the two vortices rotating rigidly and symmetrically around the origin. Linear stability analysis shows that the E point is elliptic up to the critical value $L_1^2 \sim 0.4275$ [15].
- $L_1^2 < L^2 < 1$. The $X = -\sqrt{L^2/2}$ point becomes the unstable hyperbolic point H (considering the whole system of vortices and their images, we have a configuration similar to the unstable collinear configuration of four identical vortices in the plane [16]). At the transition, two new elliptic points E_1 and E_2 arise. Their positions start from $(-\sqrt{L^2/2}, 0)$ for $L^2 = L_1^2$ and move respectively toward the origin (0,0) and the border $(-L^2, 0)$ as $L^2 \to 1$. For $L^2 < L_2^2 \sim 0.5808$ the homoclinic orbit surrounds the E_2 point. At $L^2 = L_2^2$ it crosses the origin and above the critical value it rotates around the SE point (figure 2).
- $1 < L^2 < 2$. For $L^2 > 1$ the phase space (X, Y) is no longer simply connected. Due to the constraint $X^2 + Y^2 < 1$, from the conjugation transformation we deduce that it must

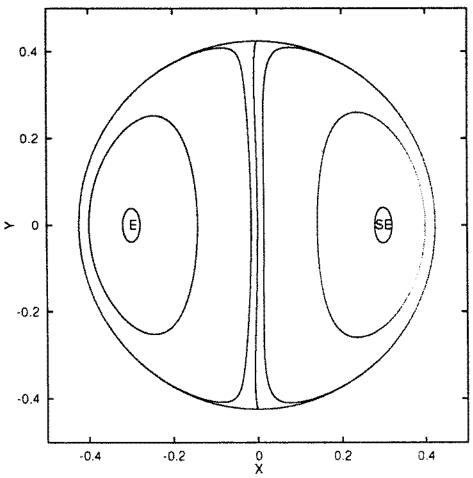

Figure 1. Vortex trajectories for the two-vortex system at $L^2 = 0.18$ in the conjugated coordinates (X, Y). E denotes the elliptic fixed point while SE represents the singular configuration of coincident vortices.

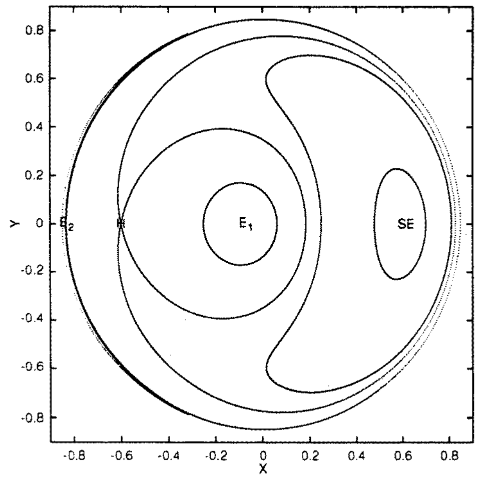

Figure 2. Vortex trajectories for $L^2 = 0.72$. H represents the fixed hyperbolic point, E_1 and E_2 are two elliptic points.

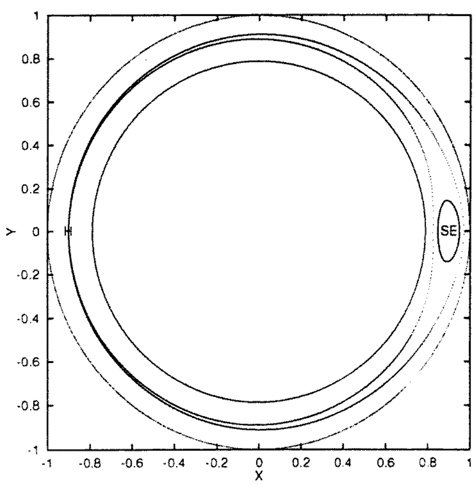

Figure 3. Vortex trajectories for $L^2 = 1.62$. The motion is constrained in the annulus $L^2 - 1 \leqslant (X^2 + Y^2) \leqslant 1$.

also be $X^2 + Y^2 > L^2 - 1$. The elliptic points E_1 and E_2 thus disappear and we have the simple topology shown in figure 3 with the hyperbolic fixed point H only. For $L^2 \to 2$ the phase space shrinks to the singular line $X^2 + Y^2 = 1$ on which the vortices collapse with their images.

The classification given above can easily be extended to a couple of vortices of arbitrary circulations giving a similar picture [17].

4. Advection of passive particles

We now discuss the motion of the advected passive particle described by the Hamiltonian H_p (8). As discussed in the appendix, we can reduce our investigation on the (P, Q) plane by taking the section at $Q_1 = 0$. In terms of physical coordinates of vortices the Poincaré plane is given by the condition

$$x_2 y_1 = y_2 x_1 \tag{10}$$

corresponding to positions collinear with the origin. As for the Eulerian vortex problem, we introduce a new set of variables

$$x = \sqrt{2P}\cos(Q) = \frac{x_3 x_2 + y_3 y_2}{r_2}$$
$$y = \sqrt{2P}\sin(Q) = \frac{y_3 x_2 - x_3 y_2}{r_2} \tag{11}$$

which will be plotted at the intersections of the vortex trajectory with the Poincaré section.

The Poincaré map will thus represent the coordinate of the passive marker relative to the position of the second vortex computed when the two vortices have zero relative angle.

Let us first consider a vortex configuration at $L^2 = 0.18$. If we place the Eulerian system in the elliptic point E (which corresponds to a local minimum for the two-vortex energy H_v, see figure 1) we obtain a stationary velocity field in the reference frame rotating with the vortices at angular velocity

$$\Omega_0 = \frac{4 + 3L^4}{2\pi L^2(4 - L^4)} \sim 0.913\,. \tag{12}$$

The Lagrangian motion in the rotating reference frame is governed by a two-dimensional autonomous Hamiltonian and it is thus always integrable. In figure 4 we present some trajectories of the passive particle for this situation. Points se_1 and se_2 represent vortex positions which are singular, stable, location for the particle. The genuine elliptic points e_1 and e_2 correspond to quasi-triangular configurations (which are known in general to be stable [16]), while the hyperbolic fixed points h_1, h_2 and h_3 correspond to unstable collinear configurations. The hyperbolic points h_2 and h_3 survive until $L^2 \sim 0.3901$ when they reach the boundary and then disappear.

If we now place the Eulerian system of two vortices at a slightly higher energy on a periodic trajectory close to the elliptic point E we obtain for the Lagrangian motion a periodic perturbation of the previous stationary Hamiltonian system which can develop chaotic behaviour.

An example of such a situation is given in figure 5 which is the Poincaré map for several passive particles. As we expected, chaotic motion develops in regions close to the hyperbolic fixed point of the unperturbed system, while the regular trajectories around the elliptic points (and the singular points) are preserved. A careful inspection of figure 5 shows large deformed islands around elliptic points e_1 and e_2, and a chain of islands inside the external chaotic region. We were not able to find any broken trajectories around the vortices (se points): this is probably due to the fact that the period for the motion of the passive next to a vortex is given approximately by $T \sim 4\pi^2 d^2$ (d being the distance from the vortex) and is thus much smaller than the perturbation period T_P (~ 8.11 in this case) up to a very small distance from the unperturbed separatrix (of the order of 10^{-6}). Broken trajectories, if any, should thus be confined to an extremely thin layer quite far from vortices.

In order to obtain a strongly chaotic dynamics for the Lagrangian motion we have to move to a higher value of the angular momentum. The large trajectory around the SE point in figure 2 is a good candidate because the relative trajectory of one vortex visits a large part of the available phase space. The modulation of the relative distance between the two vortices is comparable with the constant distance d_0 in the integrable configuration and in this case the two vortices lie alternatively on the same and on the opposite half of the domain with a period $T_P \sim 18.62$.

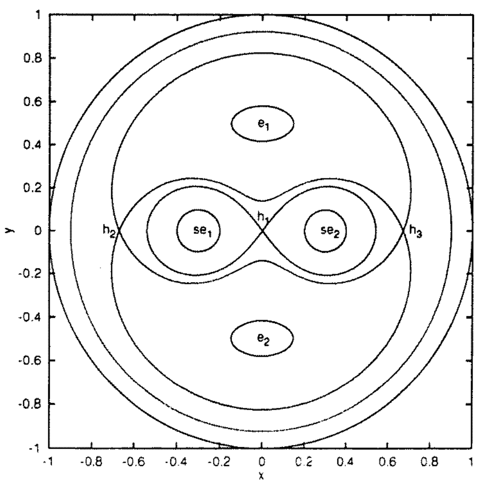

Figure 4. Passive particle Poincaré map in variables (x, y) for vortex configuration at the fixed point E for $L^2 = 0.18$. Points se_1 and se_2 represent the (singular) positions of the vortices. e_1 and e_2 are elliptic points corresponding to quasi-triangular configurations. The hyperbolic points h_1, h_2 and h_3, on collinear configurations, are connected by two different orbits.

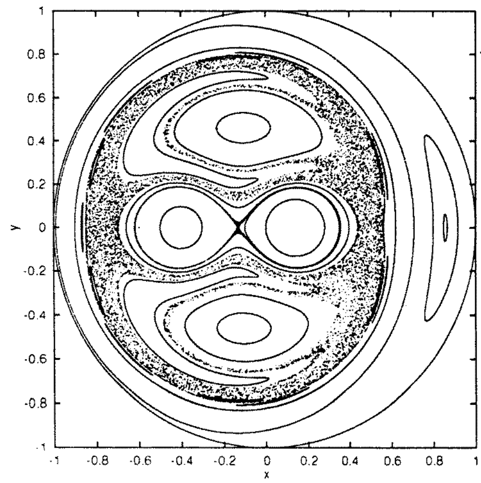

Figure 5. Passive particle Poincaré section for a vortex configuration on the large trajectory around point E at $L^2 = 0.18$.

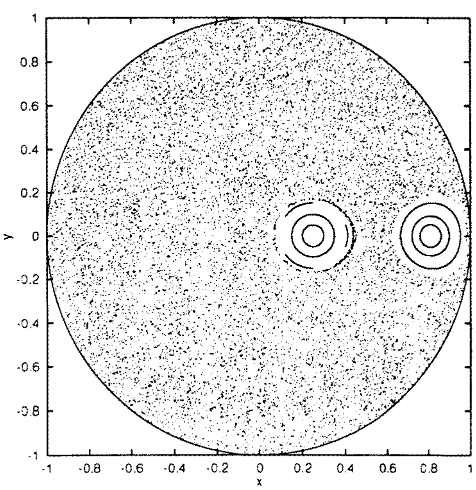

Figure 6. Lagrangian Poincaré map for a vortex configuration at $L^2 = 0.72$ on the large trajectory around the singular point SE. The 10^4 points in the large chaotic sea belong to a single trajectory. Only the regular trajectories close to vortices survive in this far-from-integrable situation.

The Lagrangian Poincaré map (figure 6) confirms the strong chaos situation expected. A single chaotic trajectory now invades almost all the phase space destroying all the regular trajectories around the unperturbed elliptic points. The only surviving regular Lagrangian regions are those around the singular vortex positions. Although we observe some broken trajectories (one of period 3 is shown around the vortex on the left), most of the regular trajectories are preserved.

Regular islands around vortices are one of the most important features of Lagrangian advection in point vortex systems. They seem to be universal, i.e. independent of the domain and on the motion (regular or chaotic) of vortices [1]. Several other simulations

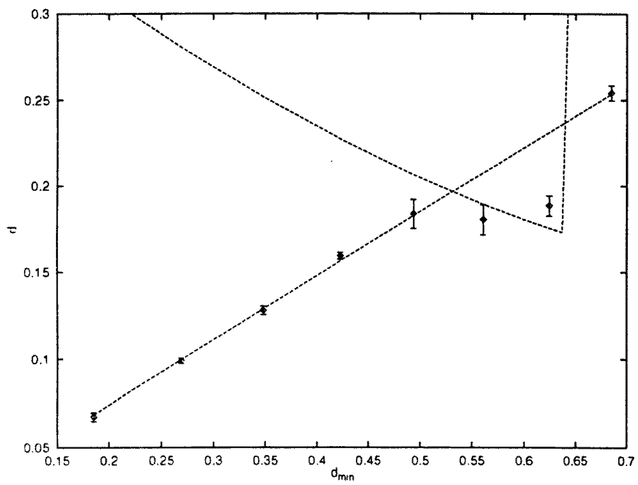

Figure 7. Linear size d of the regular island around vortices as a function of the minimal vortex distance $d_{\min}$. Broken line: linear fit $d = 0.37d_{\min}$. Broken curve: minimal distance from boundary.

showed that regular islands are always present, suggesting the physical interpretation of an impermeable barrier around each vortex which cannot be crossed by passive particles. This leads to a natural definition for the 'radius' of a point vortex from a Lagrangian point of view.

For a quantitative investigation of the vortex radius, we computed the typical linear size $d = \sqrt{A/\pi}$ of the regular island by numerical evaluation of the area A unfilled by the chaotic trajectory (see figure 6). The computation was done at $L^2 = 0.72$ for several values of the point vortex energy (7), i.e. several trajectories of figure 2. The minimal vortex distance between vortices $d_{\min}(H, L^2)$ can be computed analytically by (7) and (9). The result of our simulations is given in figure 7 where the error bars are due to the finite sampling (8000 points) in the Poincaré sections.

Regular islands are preserved ($d > 0$) for whatever minimal distance between vortices, thus no genuine vortex merging is observable from this point of view. This is not surprising, as two fundamental ingredients for merging, i.e. spatially extended vorticity distribution and dissipation, are missed in our model.

What is surprising is the strong linear correlation between d and the minimal vortex distance $d_{\min}$, given by the fit $d = 0.37d_{\min}$ (broken line). Observe that $d_{\min}$ varies over a rather large interval. The two points missed by the linear fit are due to the presence of the boundary. The radius of the regular area cannot be greater than the minimal distance between the vortex and the boundary. We found that indeed in these two cases the radius d is given, within the errors, by the minimal distance from the boundary (broken curve).

We conjecture that in a similar situation of strong Lagrangian chaos, as in the case of many advecting vortices, there still exists a region of regular Lagrangian motion surrounding each vortex whose dimension depends linearly on the minimal distance between vortices and not on the details of vortex motion.

5. Conclusions

We demonstrate that the motion of a passive particle advected by two point vortices in a circular domain is always regular around vortex positions, independently of the (regular) vortex motion. As already discussed in [1], we suggest that any system of point vortices presents the same Lagrangian properties in the neighbourhood of one vortex, because there the particle dynamics is dominated by the velocity field generated by the vortex which leads to a closed, almost circular trajectory. The influence of other vortices can be assumed as a perturbation of the integrable case. We found that the dimension of regular islands is related to the minimal distance between vortices during the dynamics, an Eulerian quantity analytically computable in the quasi-periodic motion of two vortices. Regular Lagrangian islands thus seem to be a property of point vortices which provides them with a natural and dynamical spatial extension. It would be interesting to investigate quantitatively the dimension of regular islands for a chaotic Eulerian dynamics generated by many vortices. An analytic estimate would also be welcome.

Acknowledgments

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Appendix

Here we introduce the canonical transformation used to reduce the three-vortex Hamiltonian (4) to a two-degree-of-freedom system by taking into account the conserved angular momentum L^2. The canonical transformations are similar to those introduced for studying a four-vortex problem on the plane [16] but they are valid for any set of circulations, thus allowing one to study the motion of a passive particle.

By introducing the set of canonical conjugated variables (p_i, q_i)

$$x_i = \sqrt{2p_i/\Gamma_i}\cos(q_i) \qquad y_i = \sqrt{2p_i/\Gamma_i}\sin(q_i) \tag{A1}$$

the angular momentum is given by $L^2 = 2(p_1 + p_2 + p_3)$. We now introduce the new conjugated variables (P_i, Q_i)

$$\begin{aligned} P_1 &= p_1 & Q_1 &= q_1 - q_2 \\ P_2 &= p_1 + p_2 + p_3 & Q_2 &= q_2 \\ P_3 &= p_3 & Q_3 &= q_3 - q_2 \end{aligned} \tag{A2}$$

in which $P_2 = L^2/2$ is a constant of motion and hence the conjugated Q_2 will not appear in the new Hamiltonian [18]. For simplicity of notation we restrict ourselves to the case $\Gamma_1 = \Gamma_2 = 1$ and $\Gamma_3 = \Gamma$. The new Hamilton function can be written in the form

$$H = H_0(P_1, Q_1, P_2, P_3) + \Gamma H_1(P_1, Q_1, P_2, P_3, Q_3) + \Gamma^2 H_2(P_3) \tag{A3}$$

the first term H_0 describes the dynamics in the (P_1, Q_1) plane of the two vortices Γ_1 and Γ_2. The O(Γ) term H_1 is due to the interactions of the small vortex Γ with the two unit vortices, whilst the last term H_2 results from the interaction of the Γ-vortex with its own image. In the limit $\Gamma = 0$, H becomes the one-degree-of-freedom Hamiltonian for the two-vortex motion which is thus always integrable.

The explicit expressions are

$$\begin{aligned} H_0 = &-\frac{1}{4\pi}\log\left[\frac{2(P_2 - P_3) - 4\sqrt{P_1(P_2 - P_1 - P_3)}\cos Q_1}{1 + 4P_1(P_2 - P_1 - P_3) - 4\sqrt{P_1(P_2 - P_1 - P_3)}\cos Q_1}\right] \\ &+\frac{1}{4\pi}\log[1 - 2P_1] + \frac{1}{4\pi}\log[1 - 2(P_2 - P_1 - P_3)] \end{aligned} \tag{A4}$$

$$H_1 = -\frac{1}{4\pi}\log\left[\frac{2(P_1+P_3/\Gamma)-4\sqrt{P_1P_3/\Gamma}\cos(Q_1-Q_3)}{1+4P_1P_3/\Gamma-4\sqrt{P_1P_3/\Gamma}\cos(Q_1-Q_3)}\right] - \frac{1}{4\pi}\log\left[\frac{2(P_2-P_1-P_3)+2P_3/\Gamma-4\sqrt{(P_2-P_1-P_3)P_3/\Gamma}\cos Q_3}{1+4(P_2-P_1-P_3)P_3/\Gamma-4\sqrt{(P_2-P_1-P_3)P_3/\Gamma}\cos Q_3}\right] \quad \text{(A5)}$$

and

$$H_2 = \frac{1}{4\pi}\log[1-2P_3/\Gamma]\,. \quad \text{(A6)}$$

Expression (A3) describes the motion of the three vortices in four-dimensional phase space (P_1, Q_1, P_3, Q_3) for which coordinates are proportional (through a factor $\sqrt{\Gamma_i}$) to the physical polar coordinates. The canonical variables introduced in (A2) are independent of the circulation of the three vortices and allow one to reduce the dimensionality of the system in general.

The dimensionality of the phase space can be further reduced by looking at the intersections of the trajectory with the Poincaré section [13] defined by $Q_1 = 0$ and $\dot{Q}_1 > 0$. The trajectory is thus reduced to a set of points (P_3, Q_3) which can be easily plotted and analysed.

For studying the limit $\Gamma \to 0$ we introduce the scaled variable $(P = P_3/\Gamma, Q = Q_3)$ representing the coordinate of the passive particle (see (A1) and (A2)). In this limit, expanding equations (A4)–(A6), we obtain

$$\begin{aligned} H_0 = &-\frac{1}{4\pi}\log\left[\frac{2P_2-4\sqrt{P_1(P_2-P_1)}\cos Q_1}{1+4P_1(P_2-P_1)-4\sqrt{P_1(P_2-P_1)}\cos Q_1}\right] \\ &+\frac{1}{4\pi}\log[1-2P_1]+\frac{1}{4\pi}\log[1-2(P_2-P_1)]+\mathrm{O}(\Gamma) \\ \equiv\ & H_v(P_1, Q_1, P_2)+\mathrm{O}(\Gamma) \end{aligned} \quad \text{(A7)}$$

$$\begin{aligned} H_1 = &-\frac{1}{4\pi}\log\left[\frac{2(P_1+P)-4\sqrt{P_1P}\cos(Q_1-Q)}{1+4P_1P-4\sqrt{P_1P}\cos(Q_1-Q)}\right] \\ &-\frac{1}{4\pi}\log\left[\frac{2(P_2-P_1)+2P-4\sqrt{(P_2-P_1)P}\cos Q}{1+4(P_2-P_1)P-4\sqrt{(P_2-P_1)P}\cos Q}\right]+\mathrm{O}(\Gamma) \\ \equiv\ & H_p(P, Q, P_1, Q_1, P_2)+\mathrm{O}(\Gamma) \end{aligned} \quad \text{(A8)}$$

$$H_2 = \frac{1}{4\pi}\log[1-2P]+\mathrm{O}(\Gamma)\,. \quad \text{(A9)}$$

The equations of motion for (P_1, Q_1) are given by

$$\dot{P}_1 = \frac{\partial H}{\partial Q_1} = \frac{\partial H_v}{\partial Q_1}+\mathrm{O}(\Gamma) \qquad \dot{Q}_1 = -\frac{\partial H}{\partial P_1} = -\frac{\partial H_v}{\partial P_1}+\mathrm{O}(\Gamma) \quad \text{(A10)}$$

and thus for $\Gamma \to 0$, $H_v(P_1, Q_1, P_2)$ becomes the Hamiltonian of the Eulerian motion of the vortices. The equations of motion for the passive particle are

$$\dot{P}_3 = \Gamma\dot{P} = \frac{\partial H}{\partial Q_3} = \Gamma\frac{\partial H_p}{\partial Q}+\mathrm{O}(\Gamma^2) \qquad \dot{Q}_3 = \dot{Q} = -\frac{\partial H}{\partial P_3} = -\frac{\partial H_p}{\partial P}+\mathrm{O}(\Gamma) \quad \text{(A11)}$$

and hence the time-dependent Hamiltonian $H_p(P, Q, P_1(t), Q_1(t), Q_2(t))$ governs the motion of the passive particle coordinates (P, Q).

The motion of three vortices in the disk has thus been reduced (in the limit of one vanishing circulation) to the integrable two-vortex motion in the (P_1, Q_1) plane which drives the passive particle motion in the (P, Q) plane.

References

[1] Babiano A, Boffetta G, Provenzale A and Vulpiani A 1994 *Phys. Fluids* A **6** 2465
[2] Benzi R, Patarnello S and Santangelo P 1987 *Europhys. Lett.* **3** 811
[3] Weiss J B and McWilliams J C 1993 *Phys. Fluids* A **5** 608
[4] Aref H and Pomphrey N 1980 *Phys. Lett.* **78A** 297
[5] Meleshko V V, Konstantinov M Yu, Gurzhi A A and Konovaljuk T P 1992 *Phys. Fluids* A **4** 2779
[6] Zannetti L and Franzese P 1994 *Physica* **76D** 99
[7] Meleshko V V 1994 *Phys. Fluids* **6** 6
[8] Toroczkai Z, Pentek A and Tél T 1995 *J. Phys. A: Math. Gen.* **28** 2191
[9] Novikov E A 1975 *Sov. Phys.–JETP* **41** 937
[10] Aref H 1979 *Phys. Fluids* **22** 393
[11] Aref H 1983 *Rev. Fluid Mech.* **15** 345
[12] Lin C C 1941 *Proc. Natl Acad. Sci., USA* **27** 570
[13] Lichtenberg A J and Lieberman M A 1983 *Regular and Stochastic Motion* (Berlin: Springer)
[14] Boffetta G, Paladin G and Vulpiani A 1995 *J. Phys. A: Math. Gen.* submitted
[15] Hess G B 1967 *Phys. Rev.* **161** 189
[16] Aref H and Pomphrey N 1982 *Proc. R. Soc.* A **380** 359
[17] Franzese P 1994 *Heat Transfer Enhancement by Lagrangian Chaos and Turbulence* ed H Peerhossaini and A Provenzale (Eurotherm Seminar)
[18] Arnold V I 1978 *Mathematical Methods of Classical Mechanics* (Berlin: Springer)